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AVERAGE ABSORPTION COEFFICIENTS AND MEAN FREE PATHS OF RADIATION OF THE CONTINUOUS SPECTRUM IN THE REGION OF MULTIPLE AND TOTAL IONIZATION

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Consideration is given to problems of averaging absorption coefficients of atoms and multiply charged ions with passage to the region of total ionization. It is shown that correct allowance for ionization of hydrogen-like ions makes it possible to pass automatically to coefficients of absorption in the continuous spectrum. A comparison is made with detailed calculations of mean free paths for a number of elements.

Absorption coefficients and radiation mean free paths averaged over the frequency that can be used in the region of multiple and total ionization of the gas are frequently needed for practical applications. Works [1, 2] give a derivation of the corresponding formulas for the region of multiple ionization (in the approximation of a gas that consists of ions of the same sort with the average charge $\overline{m}$); however, in the limiting case of total ionization, they do not become the known formulas for a totally ionized gas [2]*. We will show below that a more correct consideration leads to results the are suited to the entire region of temperatures, both for a multiply ionized gas and for a totally ionized gas.

The absorption coefficient $\kappa_\nu^{(m)}$ of a gas that includes the coefficients of photoabsorption $\kappa_{\nu 1}^{(m)}$ and bremsstrahlung absorption $\kappa_{\nu 2}^{(m)}$ of radiation of frequency ν by ions of multiplicity $m - 1$ ($m = 1$ corresponds to atoms) and free electrons in the field of ions of multiplicity m can be written in the Kramers approximation as [2]

$$\kappa_\nu^{(m)} = \kappa_{\nu 1}^{(m)} + \kappa_{\nu 2}^{(m)} =$$

$$= \frac{2\sqrt{2\pi}}{3\sqrt{3 m_e kT}} \frac{h^4 e^2}{\pi^2 m_e^2 c} \frac{N_m N_e}{(kT)^3} J_m^{\mathrm{H}} \left(\frac{kT}{h\nu}\right)^3 [(\exp \Psi_m - 1) + 1] =$$

$$= \bar{\kappa}_0 N_m N_e J_m^{(\mathrm{H})} \left(\frac{kT}{h\nu}\right)^3 \exp \Psi_m ; \quad \bar{\kappa}_0 = \frac{2\sqrt{2\pi}\, h^4 e^2}{3\sqrt{3 m_e kT}\, \pi^2 m_e^2 c\, (kT)^3} , \tag{1}$$

$$\Psi_m = \frac{h\nu}{kT} \Theta \left(\frac{J_m}{kT} - \frac{h\nu}{kT}\right) + \frac{J_m}{kT} \Theta \left(\frac{h\nu}{kT} - \frac{J_m}{kT}\right) ; \quad J_m^{(\mathrm{H})} = J_m m^2 ;$$

$$\Theta(x) = \begin{cases} 0 , & x < 0 , \\ 1 , & x > 0 . \end{cases}$$

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* The similar but more sophisticated problem of refining the formulas for the mean free path that characterizes the emittance of a gas in multiple (in the approximation of the average charge $\overline{m}$) and total ionization is touched upon in [3]. The formulas themselves coincide accurate to a factor of about 1.5-2. Here we are dealing with a different matter – a disagreement between the corresponding formulas for the Rosseland mean free path (and the absorption coefficient) by a factor of about 10^2.

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Here N_m is the concentration of ions of multiplicity m; N_e is the concentration of electrons; J_m is the ionization potential of the $(m-1)$-th ion; $J_m^{(\mathrm{H})} = J_{\mathrm{H}} m^2$ is the ionization potential in the hydrogen-like approximation; the remaining notation is standard.

In writing (1), in the coefficient of photoabsorption by ions of multiplicity $m-1$ ($\kappa_{\nu 1}^{(m)}$), the concentration of these ions is replaced by the concentration of ions of multiplicity m using the Saha formula

$$\frac{N_m N_e}{N_{m-1}} = 2\frac{u_m}{u_{m-1}}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(-\frac{J_m}{kT}\right), \quad m = 1, \ldots, z, \tag{2}$$

which enables us to write the result in a more compact form – both coefficients now depend similarly on the concentration of the particles. In this connection, we note that an approximate formula that is correct only for $N_{m-1} \gg N_m$ (a low degree of ionization of the ions of the sort in question) was immediately used for $\kappa_{\nu 1}^{(m)}$ in [1]. Hence disagreements with the exact formula for κ_ν arise in passage to total ionization.

By introduction of the function Ψ_m in (1) it is allowed for that with a quantum energy $h\nu$ that is larger than the ionization potential J_m the number of levels involved in photoabsorption becomes fixed. In calculating $\kappa_{\nu 1}^{(m)}$, summation over the levels is replaced, as usual [2], by integration, although several lower levels can, in principle, be left in the form of a sum for more accurate determination of $\kappa_{\nu 1}^{(m)}$ [3, 4].

To obtain the total absorption coefficient, expression (1) should be summed over the sorts of ions with account for the Saha formulas (2):

$$\kappa_\nu = \sum_{m=1}^{z} \kappa_\nu^{(m)}, \tag{3}$$

where z is the charge of the nucleus of the element in question. If we pass further to a thermodynamic model of a gas that consists of ions of one sort with the average charge $\overline{m}$, then in all the sums over the sorts m of ions that occur in different kinds of calculations one term each that corresponds to the average charge $\overline{m}$ remains; precisely this approximation is employed in [1, 2]. Here, for simplicity, the subsequent results will also be given in this approximation. Then, instead of (3), we have

$$\kappa_\nu = \kappa_\nu^{(\overline{m})} = \overline{\kappa}_0 N_{\overline{m}} N_e J_{\overline{m}}^{(\mathrm{H})} \left(\frac{kT}{h\nu}\right)^3 \exp \Psi_{\overline{m}}, \quad J_{\overline{m}}^{(\mathrm{H})} = J_{\mathrm{H}} \overline{m}^2. \tag{4}$$

When $\overline{m} = z$, (4) becomes the formula for single ionization if $N_{\overline{m}}$ means the number of ions in the gas and (2) is allowed for. In general, according to the model we have $N_{\overline{m}} = N$, where N is the total number of heavy particles of the gas, which should be substituted into (4) when $\overline{m} > 1$; with $\overline{m} = z$ formula (4) describes the behavior of a totally ionized gas.

In describing transfer of energy by radiation in the approximation of radiant heat conduction (large optical thicknesses of the plasma), we must have the radiation mean free path l_{R} averaged according to Rosseland. As is known [2], it is determined by the formula

$$l_{\mathrm{R}} = \int_0^\infty \frac{1}{\kappa_\nu(u)} G'(u)\, du; \quad G'(u) = \frac{15}{4\pi^4}\frac{u^4 \exp(-u)}{(1-\exp(-u))^3}; \quad u = \frac{h\nu}{kT}. \tag{5}$$

By employing (5), (3), and (1) we find

$$l_{\mathrm{R}} = \frac{15}{\pi^4}\frac{1}{\overline{\kappa}_0 N_e} \int \frac{du\, u^7 \exp(-u)}{(1-\exp(-u))^3} \frac{1}{\sum\limits_m N_m J_m^{(\mathrm{H})} \exp \Psi_m}, \tag{6}$$

from which, in the average-charge approximation, it follows that

$$l_{\mathrm{R}} = \frac{15}{4\pi^4 \kappa_0} \left\{ \int_0^{\bar{u}_*} \frac{du u^7 \exp(-2u)}{(1-\exp(-u))^3} + \int_{\bar{u}_*}^{\infty} \frac{du u^7 \exp(-(u+u_*))}{(1-\exp(-u))^3} \right\}, \tag{7}$$

$$\kappa_0 = \bar{\kappa}_0 N N_{\mathrm{e}} J_{\overline{m}}^{(\mathrm{H})} ; \quad \bar{u}_* = \bar{J}/kT ; \quad \bar{J} = J_{\overline{m}} .$$

The ultimate result in (7) depends on $\bar{u}_*$. Since the maxima of the integrands in (7) are close to $u_1 \approx 3.5$ and $u_2 \approx 7$, respectively, in the first and second integrals we should distinguish three cases for which approximate integration can be performed: 1) $\bar{u}_* \leq 1$; 2) $3.5 < \bar{u}_* < 7$; 3) $\bar{u}_* > 7$. For them, we have

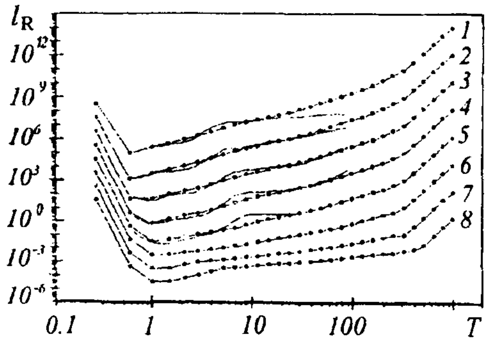
$$l_{\mathrm{R}} = \begin{cases} l_{\mathrm{R}}^{(1)} \approx \dfrac{15}{4\pi^4 \kappa_0} 7\,!\exp(-\bar{u}_*) = l_{\mathrm{R}}^{(T)} \exp(-\bar{u}_*), & \bar{u}_* \leq 1 ; \\ l_{\mathrm{R}}^{(2)} \approx l_{\mathrm{R}}^{(T)} \left(\dfrac{1}{2^8} + \exp(-\bar{u}_*) \right) = l_{\mathrm{R}}^{(T)} (\exp(-5.5) + \exp(-\bar{u}_*)), & 3.5 < \bar{u}_* < 7 ; \\ l_{\mathrm{R}}^{(3)} \approx l_{\mathrm{R}}^{(T)} \dfrac{1}{2^8} \approx \dfrac{1}{260} l_{\mathrm{R}}^{(T)}, & \bar{u}_* > 7 . \end{cases} \tag{8}$$

In (8), $l_{\mathrm{R}}^{(T)}$ denotes the Rosseland mean free path of radiation for a totally ionized gas with the charge $\overline{m}$, where only bremsstrahlung radiation remains, in practice. As above, the expressions obtained become the formulas for a singly charged plasma if we set $\kappa_0 = \bar{\kappa}_0 N N_{\mathrm{e}} J_1$ and $\bar{J} = J_1$. It also clearly follows from (8) how the character of the dependence of l_{R} changes as the temperature increases – in passage to multiple and total ionization. Of interest is the fact that the dependence $l_{\mathrm{R}} = l_{\mathrm{R}}^{(3)}$ is mainly retained in traversal of successive steps of ionization of a gas up to $\overline{m} \sim z$ with increase in the temperature since normally $\bar{u}_* = \bar{J}/kT > 5$; only in the final step, in passage to total ionization, does $\bar{u}_*$ begin to decrease: the dependence $l_{\mathrm{R}} = l_{\mathrm{R}}^{(1)}$ is "switched on" and the coefficient before $l_{\mathrm{R}}^{(T)}$ increases rapidly from ~1/260 to ~1. The result contained in [1, 2] for multiple ionization, for the reason that was mentioned above, always coincides with $l_{\mathrm{R}}^{(3)}$ and yields no transition to $l_{\mathrm{R}}^{(1)}$.

By analogous calculations that are not given here we obtain formulas for Planck-averaged absorption coefficients, which are used for calculating the emissive power of a plasma with a small optical thickness.

We note that the formula $l_{\mathrm{R}} = l_{\mathrm{R}}^{(2)}$ can be employed as an interpolation formula when l_{R} as a function of T is described in the entire temperature interval. Together with the Raizer formulas of the thermodynamic model of the average charge and the Sommerfeld formula for atomic ionization potentials [5], we obtain a model of the optical properties of a gas that is convenient for practical evaluations and describes transfer of energy by radiant heat conduction in multiple and total ionization. We give a complete summary of the formulas required for this purpose:

$$\begin{gathered} l_{\mathrm{R}} = l_{\mathrm{R}}^{(T)} (\exp(-5.5) + \exp(-\bar{u}_*)) ; \\ l_{\mathrm{R}}^{(T)} = \frac{15 \cdot 7\,!}{4\pi^4 \kappa_0} = \frac{15 \cdot 7\,!}{4\pi^4 \bar{\kappa}_0 N N_{\mathrm{e}} J_{\overline{m}}^{(\mathrm{H})}} \cong 4.8 \cdot 10^{24} \frac{T^{7/2}}{\overline{m}^3 N^2} \ (\mathrm{cm}) ; \\ N_{\mathrm{e}} = \overline{m} N ; \quad J_m^{(\mathrm{H})} = J_{(\mathrm{H})} \overline{m}^2 ; \quad \overline{m} = \frac{A_{\mathrm{e}} T^{3/2}}{N} \exp\left(-\frac{\bar{J}}{kT} \right) ; \quad A_{\mathrm{e}} = \left(\frac{2\pi m_{\mathrm{e}} k}{h^2} \right)^{3/2} ; \end{gathered} \tag{9}$$



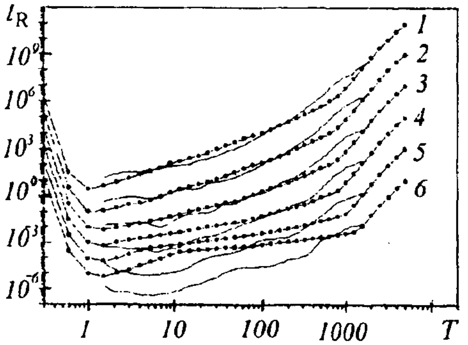
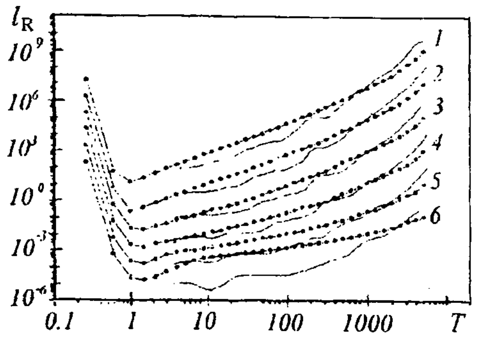


Fig. 1. Rosseland mean free paths of radiation l_R (cm) for aluminum as functions of the temperature T (eV) for the relative densities δ: 1) 10^{-4}; 2) 10^{-3}; 3) 10^{-2}; 4) 0.1; 5) 1; 6) 10; 7) 100; 8) 1000.

Fig. 2. Rosseland mean free paths of radiation l_R for titanium as functions of the temperature T for the densities ρ: 1) 10^{-4} g/cm^3; 2) 10^{-3}; 3) 10^{-2}; 4) 0.1; 5) 1; 6) 10.



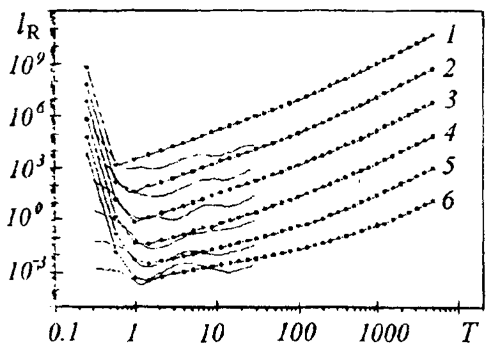


Fig. 3. Rosseland mean free paths of radiation l_R for tin as functions of the temperature T for the densities ρ: 1) 10^{-4} g/cm^3; 2) 10^{-3}; 3) 10^{-2}; 4) 0.1; 5) 1; 6) 10.

Fig. 4. Rosseland mean free paths of radiation l_R for bismuth as functions of the temperature T for the relative densities δ: 1) 10^{-4}; 2) 10^{-3}; 3) 10^{-2}; 4) 0.1; 5) 1; 6) 10.

$$\bar{u}_* = \frac{\bar{J}}{kT}\,; \quad \bar{J} = \bar{J}\left(\bar{m}\right) = \frac{dQ\left(\bar{m}\right)}{d\bar{m}}\,; \quad Q\left(\bar{m}\right) = \frac{1.27\bar{m}^{7/3}}{1 - 0.903\left(\dfrac{\bar{m}}{z}\right)^{1/4}} \ (\text{эВ}) .$$

Figures 1-4 show results of calculating l_R by formulas (9) (J_1 is used when $\bar{J} < J_1$, and J_z is used when $\bar{m} \geq z$) – the solid curves with points. Results of calculations from [6] for Al, from [7] for Ti and Sn, and from [8] for Bi are shown by the solid curves.

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NOTATION

$\bar{m}$, average charge of the plasma; ν, radiation frequency; $\kappa_\nu^{(m)}$, spectral coefficient of absorption at the frequency ν; $\kappa_{\nu1}^{(m)}$, spectral coefficient of photoabsorption at the frequency ν; $\kappa_{\nu2}^{(m)}$, spectral coefficient of bremsstrahlung absorption at the frequency ν; h, Planck constant; e, electron charge; m_e, electron mass; k, Boltzmann constant; T, plasma temperature; c, velocity of light; N_m, concentration of particles of multiplicity m; N_e, concentration of electrons in the plasma; J_m, ionization potential of a particle of multiplicity m; u_m, statistical sum of a particle of multiplicity m; l_R and $l_R^{(T)}$, Rosseland mean free paths of radiation; $G'(u)$, derivative of the

Planck function; N, concentration of particles; z, nuclear charge; ρ, density; δ, relative density ($\delta = N/N_L$; N_L, Loschmidt number).

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